

Physics 137B, Spring 2004

Problem Set # 7

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Problem 1:

$$H_0 = -\frac{\hbar^2}{2m} \frac{d}{dx^2} + \frac{1}{2} kx^2 \leftarrow \text{Harmonic Oscillator}$$

$$\begin{aligned} H'(t) &= -q \times \epsilon_0 \exp(-t/\tau) ; \quad x = \left(\frac{\hbar}{2m\omega}\right)^{1/2} (a^\dagger + a) ; \quad a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle \\ &= -q \epsilon_0 \left(\frac{\hbar}{2m\omega}\right)^{1/2} (a^\dagger + a) \exp(-t/\tau). \end{aligned}$$

at $t \leq 0$, the oscillator is in the ground state $|0\rangle$.

We are looking for the prob. of the oscillator to be in an excited state (to 1st order):

$$C_{n \neq 0}^{(1)}(t) = \frac{1}{i\hbar} \int_0^t H'_{n0}(t') \exp(i\omega_{n0} t') dt'$$

$$\omega_{n0} = \frac{E_n^{(0)} - E_0^{(0)}}{\hbar} = \frac{1}{\hbar} \left[\hbar\omega \left(n + \frac{1}{2}\right) - \frac{\hbar\omega}{2} \right] = \omega n$$

$$\begin{aligned} H'_{n0} &= \langle n | H' | 0 \rangle = -q \epsilon_0 \left(\frac{\hbar}{2m\omega}\right)^{1/2} e^{-t/\tau} \langle n | (a^\dagger + a) | 0 \rangle \\ &= -q \epsilon_0 \left(\frac{\hbar}{2m\omega}\right)^{1/2} e^{-t/\tau} [\delta_{n,1}] \end{aligned}$$

$$\begin{aligned} C_{n \neq 0}^{(1)}(t) &= \frac{-1}{i\hbar} q \epsilon_0 \left(\frac{\hbar}{2m\omega}\right)^{1/2} \delta_{n,1} \int_0^t e^{-t'/\tau} e^{i\omega n t'} dt' \\ &= i q \epsilon_0 \left(\frac{1}{2m\omega\tau}\right)^{1/2} \delta_{n,1} \int_0^t e^{i(\omega n + i/\tau)t'} dt' \\ &= i q \epsilon_0 \left(\frac{1}{2m\omega\tau}\right)^{1/2} \delta_{n,1} \frac{1}{i\alpha} \int_0^{i\alpha t} e^u du = \frac{q \epsilon_0}{(2m\omega\tau)^{1/2}} \delta_{n,1} \frac{1}{\alpha} [e^{i\alpha t} - 1] \end{aligned}$$

$$P_{n0}^{(1)}(t) = |C_{n \neq 0}^{(1)}(t)| = \frac{q^2 \epsilon_0^2}{2m\omega\tau} \frac{1}{|\alpha|^2} |e^{i\alpha t} - 1|^2$$

$$\text{We are interested in the limit } t \rightarrow \infty \quad \therefore \lim_{t \rightarrow \infty} P_{n \neq 0,0}^{(1)}(t) = \frac{q^2 \epsilon_0^2}{2m\omega\tau} \frac{1}{|\alpha|^2} |e^{i\alpha t} - 1|^2$$

$$|\alpha|^2 = \alpha^* \alpha = (\omega n - i/\tau)(\omega n + i/\tau) = (\omega n)^2 + \frac{1}{\tau^2} = \frac{(\tau \omega n)^2 + 1}{\tau^2}$$

$$\therefore \frac{1}{|\alpha|^2} = \frac{\tau^2}{1 + (\tau \omega n)^2}$$

$$\begin{aligned} |e^{i\omega t} - 1|^2 &= |e^{i\omega nt} e^{-t/\tau} - 1|^2 = |(\cos(\omega nt) + i\sin(\omega nt))e^{-t/\tau} - 1|^2 \\ &= |(e^{-t/\tau} \cos(\omega nt) - 1) + i \sin(\omega nt) e^{-t/\tau}|^2 \\ &= [(e^{-t/\tau} \cos(\omega nt) - 1) + i \sin(\omega nt) e^{-t/\tau}] [(\bar{e}^{-t/\tau} \cos(\omega nt) - 1) - i \sin(\omega nt) e^{-t/\tau}] \\ &= (e^{-t/\tau} \cos(\omega nt) - 1)^2 + \sin^2(\omega nt) e^{-2t/\tau} = e^{-2t/\tau} \cos^2(\omega nt) - 2e^{-t/\tau} \cos(\omega nt) + 1 \\ &\quad + \sin^2(\omega nt) e^{-2t/\tau} \end{aligned}$$

$$P_{n \neq 0, 0}(t) = \frac{q^2 \epsilon_0^2}{2m\omega} \frac{\tau^2}{1 + (\tau \omega n)^2} \left[1 - 2e^{-t/\tau} \cos(\omega nt) + e^{-2t/\tau} \right] \delta_{n, 1}$$

$$\lim_{t \rightarrow \infty} P_{n \neq 0, 0}(t) = \frac{q^2 \epsilon_0^2}{2m\omega} \frac{\tau^2}{1 + (\tau \omega n)^2} \quad \delta_{n, 1} = \begin{cases} \frac{q^2 \epsilon_0^2}{2m\omega} \frac{\tau^2}{1 + (\tau \omega)^2} & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 2

$$\mathcal{E}(t) = \mathcal{E}_0 / (t^2 + \tau^2)$$

at $t = -\infty$ the atom is in the ground state. We want the prob that it will be in the $2p$ states at time $t = \infty$.

$$P_{1s, 2p}^{(1)}(t) = \frac{1}{\hbar^2} \left| \int_{-\infty}^{\infty} H'_{1s, 2p} \exp(i\omega_{1s, 2p} t') dt' \right|^2$$

What is H' ? $H = -e \vec{\mathcal{E}}(t) \cdot \vec{x} = -e \mathcal{E}(t) z$ (I have chosen the direction of the electric field in the z direction for simplicity).

$$H'_{1s, 2p} = \langle 1s | H' | 2p \rangle = -e \mathcal{E}(t) \langle 1s | z | 2p \rangle$$

Notice that there are 3 $2p$ states: $|2, 1, 0\rangle, |2, 1, 0\rangle, |2, 1, 1\rangle$

The Prob $P_{1s, 2p}$ is the sum of the prob. of transitioning to each of these states.

$$\begin{aligned} \cdot \langle 100 | z | 211 \rangle &= - \left(\frac{3}{32\pi^2} \right)^{1/2} \frac{2}{\sqrt{24}} \frac{1}{a^3} \int_0^\infty \int_0^{2\pi} \int_0^\pi r^2 \sin\theta dr d\theta d\phi \left[e^{-r/a} r \cos\theta \frac{r}{a} e^{-r/2a} \sin\theta e^{i\phi} \right] \\ &= - \left(\frac{3}{32\pi^2} \right)^{1/2} \frac{2}{\sqrt{24}} \frac{1}{a^3} \int_0^\infty \frac{r^5}{a} e^{-3r/2a} dr \int_0^{2\pi} e^{i\phi} d\phi \int_0^\pi \underbrace{\sin^2\theta \cos\theta}_{11} d\theta = 0 \end{aligned}$$

$$\begin{aligned} \cdot \langle 100 | z | 21-1 \rangle &= + \left(\frac{3}{32\pi^2} \right)^{1/2} \frac{2}{\sqrt{24}} \frac{1}{a^3} \int_0^\infty \int_0^{2\pi} \int_0^\pi r^2 \sin\theta dr d\theta d\phi \left[e^{-r/a} r \cos\theta \frac{r}{a} e^{-r/2a} \sin\theta e^{-i\phi} \right] \\ &\propto \int_0^\pi \underbrace{\sin^2\theta \cos\theta}_{11} d\theta = 0 \end{aligned}$$

$$\begin{aligned} \cdot \langle 100 | z | 210 \rangle &= \left(\frac{1}{4\pi} \right)^{1/2} \left(\frac{3}{4\pi} \right)^{1/2} \frac{2}{\sqrt{24}} \frac{1}{a^4} \int_0^\infty \int_0^{2\pi} \int_0^\pi r^2 \sin\theta dr d\theta d\phi \left[e^{-r/a} r \cos\theta r e^{-r/2a} \cos\theta \right] \\ &= \frac{1}{8\pi} \frac{1}{\sqrt{8}} \frac{1}{a^4} (2\pi) \int_0^\infty r^4 e^{-3r/2a} dr \int_0^\pi \underbrace{\sin\theta \cos^2\theta}_{11/2/3} d\theta \\ &= \frac{2}{3} \frac{1}{a^4} \frac{1}{2\sqrt{2}} \int_0^\infty r^4 e^{-3r/2a} dr \quad \text{let } u = \frac{3r}{2a}, du = \frac{3}{2a} dr \\ &= \frac{2a}{3} \frac{1}{3a^4\sqrt{2}} \int_0^\infty \left(\frac{2a}{3} u \right)^4 e^{-u} du = \left(\frac{2a}{3} \right)^5 \frac{1}{3a^4\sqrt{2}} \underbrace{\int_0^\infty u^4 e^{-u} du}_{24} \\ &= \frac{2^8}{3\sqrt{2}} \left(\frac{2}{3} \right)^5 a^5 = a_0 \frac{2^{15/2}}{3^5} = a_0 \left(\frac{2^{3/2}}{3} \right)^5 \end{aligned}$$

\therefore we see that the transition is only possible to the $|210\rangle$ state.

$$P_{1s, 2p}^{(1)}(t) = \frac{1}{t^2} \left| \int_{-\infty}^{\infty} -a_0 \left(\frac{2^{3/2}}{3} \right)^5 e^{\frac{\epsilon_0}{t^2 + t'^2} t'} e^{i\omega_{1s, 2p} t'} dt' \right|^2$$

$$(i\omega_{1s, 2p} = \frac{E_2 - E_1}{\hbar} = -\frac{E_1 + E_1}{4} = \frac{3}{4} \frac{E_1}{\hbar} = \omega)$$

$$= \frac{1}{t^2} a_0^2 \left(\frac{2^{3/2}}{3} \right)^{10} e^{2\epsilon_0^2} \underbrace{\left| \int_{-\infty}^{\infty} \frac{e^{i\omega t'}}{t'^2 + \omega^2} dt' \right|^2}_{\left(\frac{\pi}{\omega} \right) e^{-\omega t}} = \left(\frac{2^{3/2}}{3} \right)^{10} \left(\frac{\pi}{\omega} \right)^2 (a_0 e \epsilon_0)^2 e^{-\frac{3}{4} \frac{E_1}{\hbar} t}$$

$$\lim_{t \rightarrow \infty} P_{1s, 2p}^{(1)}(t) = P_{1s, 2p}^{(1)}(t) = P_{1s, 2p}^{(1)} \text{ since the prob. is independent of time.}$$

Problem 3

$$c_a(t) = e^{-i\alpha t} \left[\cos(\beta t) + i \frac{\gamma}{\beta} \sin(\beta t) \right]; c_b(t) = -i \frac{H'_{ba}}{t\beta} e^{-i(\alpha - \omega_{ba})t} \sin \beta t$$

$$\dot{c}_a(t) = -i\alpha e^{-i\alpha t} \left[\cos(\beta t) + i \frac{\gamma}{\beta} \sin(\beta t) \right] + e^{-i\alpha t} \left[i \frac{\gamma}{\beta} \cos(\beta t) - \beta \sin(\beta t) \right]$$

$$\dot{c}_b(t) = -i \frac{H'_{ba}}{t\beta} \left[-i(\alpha - \omega_{ba}) e^{-i(\alpha - \omega_{ba})t} \sin \beta t + \beta e^{-i(\alpha - \omega_{ba})t} \cos \beta t \right]$$

Let's check 9.37(a) first:

$$(LHS) = it c_a = \alpha t e^{-i\alpha t} \left(\cos(\beta t) + i \frac{\gamma}{\beta} \sin(\beta t) \right) + i t e^{-i\alpha t} \left[i \frac{\gamma}{\beta} \cos(\beta t) - \beta \sin(\beta t) \right]$$

$$= (\alpha t e^{-i\alpha t} - \gamma t e^{-i\alpha t}) \cos(\beta t) + i \left(\frac{\gamma t e^{-i\alpha t}}{\beta} - \beta t e^{-i\alpha t} \right) \sin(\beta t)$$

$$(RHS) = H'_{aa} c_a(t) + H'_{ab} e^{-i\omega_{ba} t} c_b(t) = H'_{aa} e^{-i\alpha t} \left[\cos(\beta t) + i \frac{\gamma}{\beta} \sin(\beta t) \right] - i \frac{|H'_{ab}|^2}{t\beta} e^{-i\alpha t} \sin \beta t$$

Now let's figure out what H'_{aa} & $|H'_{ab}|^2$ are in terms of α, β, γ :

$$\gamma = \alpha - H'_{aa}/t \Rightarrow H'_{aa} = t(\alpha - \gamma)$$

$$\beta^2 = \frac{1}{4t^2} (H'_{bb} - H'_{aa} + t\omega_{ba})^2 + \frac{1}{t^2} |H'_{ba}|^2 = (\alpha - 2(\frac{H'_{aa}}{2t}))^2 + \frac{1}{t^2} |H'_{ba}|^2 \quad (\text{since } \alpha = \frac{1}{2t}(H'_{aa} + H'_{bb} - t\omega_{ba}))$$

$$\Rightarrow |H'_{ba}|^2 = t^2 \left[\beta^2 - (\alpha - \frac{H'_{aa}}{t})^2 \right] = t^2 \beta^2 - t^2 \left(\alpha^2 - 2\frac{\alpha H'_{aa}}{t} + \frac{H'^2_{aa}}{t^2} \right)$$

$$\therefore = t^2 \beta^2 - t^2 \alpha^2 + 2\alpha t^2 (\alpha - \gamma) - t^2 (\alpha^2 - 2\alpha \gamma + \gamma^2) = t^2 \beta^2 - t^2 \gamma^2$$

$$\therefore RHS = t(\alpha - \gamma) e^{-i\alpha t} \left[\cos \beta t + i \frac{\gamma}{\beta} \sin \beta t \right] - i \frac{[t^2 \beta^2 - t^2 \gamma^2]}{t\beta} e^{-i\alpha t} \sin \beta t$$

$$= (\alpha t - \gamma t) e^{-i\alpha t} \cos \beta t + i \left[\frac{\gamma \alpha t}{\beta} - \frac{t \gamma^2}{\beta} - t \beta + t \frac{\gamma^2}{\beta} \right] e^{-i\alpha t} \sin \beta t$$

$$= (\alpha t - \gamma t) e^{-i\alpha t} \cos \beta t + i \left[\frac{\gamma \alpha t}{\beta} - \beta t \right] e^{-i\alpha t} \sin \beta t = it c_a(t) \checkmark$$

Now, let's check 9.37(b):

$$\begin{aligned} LHS: it c_b(t) &= it \left[-i \frac{H'_{ba}}{t\beta} (-i(\alpha - \omega_{ba}) e^{-i(\alpha - \omega_{ba})t} \sin \beta t + \beta e^{-i(\alpha - \omega_{ba})t} \cos \beta t) \right] \\ &= -i \frac{H'_{ba}}{\beta} (\alpha - \omega_{ba}) e^{-i(\alpha - \omega_{ba})t} \sin \beta t + H'_{ba} e^{-i(\alpha - \omega_{ba})t} \cos \beta t \end{aligned}$$

RHS: $H'_{ba} e^{i\omega_{ba}t} c_a(t) + H'_{bb} c_b(t)$.

First let's figure out H'_{bb} : $\alpha = \frac{1}{2t} (H'_{aa} + H'_{bb} + t\omega_{ba})$

$$\Rightarrow H'_{bb} = 2t\alpha - H'_{aa} - t\omega_{ba} = 2t\alpha - t\alpha + t\gamma - t\omega_{ba} = t(\alpha + \gamma - \omega_{ba})$$

$$\begin{aligned}\therefore \text{RHS: } & H'_{ba} e^{i\omega_{ba}t} \left[e^{-i\alpha t} \left(\cos \beta t + i \frac{\gamma}{\beta} \sin \beta t \right) \right] + t(\alpha + \gamma - \omega_{ba}) \frac{i}{\hbar \beta} H'_{ba} e^{-i(\alpha - \omega_{ba})t} \sin \beta t \\ &= H'_{ba} e^{-i(\alpha - \omega_{ba})t} \cos \beta t - i \left[\frac{\alpha + \gamma - \omega_{ba}}{\beta} - \frac{\gamma}{\beta} \right] H'_{ba} e^{-i(\alpha - \omega_{ba})t} \sin \beta t \\ &= H'_{ba} e^{-i(\alpha - \omega_{ba})t} \cos \beta t - i \left[\frac{\alpha - \omega_{ba}}{\beta} \right] H'_{ba} e^{-i(\alpha - \omega_{ba})t} \sin \beta t = i \hbar c_b(t) \checkmark\end{aligned}$$

Problem 4

Exact solutions: $c_a(t) = e^{-i\alpha t} \left[\cos \beta t + i \frac{\gamma}{\beta} \sin \beta t \right]$

$$c_b(t) = -i \frac{H'_{ba}}{\hbar \beta} e^{-i(\alpha - \omega_{ba})t} \sin \beta t$$

Approximate to first order in H' & small time t .

- $c_a(t)$: Since t is small $\cos \beta t = 1 - \frac{1}{2} (\beta t)^2 + O^4(t)$
 $\sin \beta t = \beta t - \frac{1}{6} (\beta t)^3 -$
 $e^{-i\alpha t} = 1 - i\alpha t - \frac{1}{2} \alpha^2 t^2 -$

$$c_a(t) = (1 - i\alpha t - \frac{1}{2} \alpha^2 t^2 + O^4(t)) \left[(1 - \frac{1}{2} (\beta t)^2 + O^4(t)) + i \frac{\gamma}{\beta} \left(\beta t - \frac{1}{6} (\beta t)^3 + O^5(t) \right) \right]$$

We need to keep to first order in H' . ∵ First order in α, β , & γ .

$$= (1 - i\alpha t + O^2(\alpha)) \left[(1 + O^2(\beta)) + i \frac{\gamma}{\beta} (\beta t + O^3(\beta)) \right]$$

$$= (1 - i\alpha t + O^2(\alpha)) [1 + i\gamma t + O^2(\beta)] = 1 + i\gamma t - i\alpha t + O^2(\alpha)$$

$$= 1 - i(\alpha - \gamma)t = 1 - i \frac{H'_{aa} t}{\hbar} \approx e^{-i \frac{H'_{aa} t}{\hbar}} \checkmark$$

$$\bullet \underline{C_b(t)}: \text{Exact answer: } -i \frac{\hbar b_a}{\hbar \beta} e^{-i(\alpha - \omega_{ba})t} \sin \beta t = C_b(t)$$

$$\therefore C_b(t) = -i \frac{\hbar b_a}{\hbar \beta} [1 - i(\alpha - \omega_{ba})t + O^2(\alpha)] [\beta t + O^3(\beta)]$$

$$= -i \frac{\hbar b_a}{\hbar} [t - i(\alpha - \omega_{ba})t^2 + O^2(\alpha)] \quad (\text{Now bring in the } i).$$

$$= \frac{\hbar b_a}{\hbar} [-it - (\alpha - \omega_{ba})t^2 + O^2(\alpha)] \quad (\text{Now multiply by } 1 = \frac{\omega_{ba}}{\omega_{ba}})$$

$$= \frac{\hbar b_a}{\hbar \omega_{ba}} [-i\omega_{ba}t - \underbrace{\omega_{ba}(\alpha - \omega_{ba})t^2}_{\text{This is second order in } H'} \dots]$$

$$C_b(t) = \frac{\hbar b_a}{\hbar \omega_{ba}} [-i\omega_{ba}t + O^2(\alpha)] = \frac{\hbar b_a}{\hbar \omega_{ba}} (1 - e^{i\omega_{ba}t}) \checkmark$$